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LETTER TO THE EDITOR

Generalized deformed oscillator and nonlinear algebras

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Abstract. The harmonic oscillator is deformed arbitrarily and the properties of the deformed oscillator algebra are studied. The eigenstates and eigenvalues of the deformed oscillator are calculated. This generalized deformed oscillator algebra is used to create nonlinear deformation of the classical SU(2) algebra. The deformed boson realizations of an arbitrary nonlinear SU(2) algebra are calculated. The algebra of the q-deformed oscillator is a special case of the generalized deformed oscillator algebra.

The quantum group $SU_q(2)$ was introduced by Kulish and Reshetikhin [1]. Biedenharn [2] introduced the q-deformed harmonic oscillator and constructed a realization of the $SU_q(2)$, this was done independently by Macfarlane [3]. Polychronakos [4] studied the necessary conditions for the existence of a generalized deformed quantum group with co-multiplication; in this paper the author concluded that the only deformation that can be obtained through an oscillator realization is the quantum q-deformation. In this letter we start from an arbitrary deformation of the oscillator and we construct all the general deformed oscillator algebra studying its properties. These results are general and they can be applied for any deformed case, including the q-deformed one.

A general deformation of the harmonic oscillator can be given by the basic relation

$$aa^\dagger = g(a^\dagger a) \tag{1}$$

where a and $a^\dagger$ are Hermitian conjugate operators. In the ordinary oscillator algebra the function $g(x)$ is defined by

$$g(x) = 1 + x \tag{2}$$

which leads to the commutation relation

$$[a, a^\dagger] = 1. \tag{3}$$

For the q-deformed oscillator the corresponding function $g(x)$ is given by

$$g(x) = \sqrt{1 - x^2 \sin^2 \tau} + x \cos \tau. \tag{4}$$

This formula will be discussed later.

The *number* operator N, by definition, satisfies the commutation relations:

$$[a, N] = a \qquad \text{and} \qquad [a^\dagger, N] = -a^\dagger. \tag{5}$$

We assume that this operator is given by a relation as follows:

$$N = f(a^\dagger a). \tag{6}$$

The function $f(x)$ must be related to the function $g(x)$, as we shall show.

If equation (1) is true then by induction the following relations can be proved:

$$[a, (a^\dagger a)^n] = ((g(a^\dagger a))^n - (a^\dagger a)^n)a \tag{7}$$

and

$$[a^\dagger, (a^\dagger a)^n] = -a^\dagger((g(a^\dagger a))^n - (a^\dagger a)^n). \tag{8}$$

For a function $f(x)$ holomorphic near to zero, equations (7) and (8) imply that

$$[a, f(a^\dagger a)] = (f(g(a^\dagger a)) - f(a^\dagger a))a \tag{9}$$

and

$$[a^\dagger, f(a^\dagger a)] = -a^\dagger(f(g(a^\dagger a)) - f(a^\dagger a)). \tag{10}$$

If the function $f(x)$ is chosen such that

$$f(g(x)) = 1 + f(x) \tag{11}$$

then the commutation relations (5) are satisfied. In worked examples we can see that, from equation (11) and the function $g(x)$, the function $f(x)$ can be determined.

Usually the inverse problem is posed; the function $F(x)$ is given:

$$F = f^{-1} \qquad \text{or} \qquad F(f(x)) = x \tag{12}$$

and the function $g(x)$ is determined as follows:

$$g(x) = F(1 + f(x)) \tag{13}$$

In this case equations (1), (5) and (6) are valid.

In equation (11) if x is replaced by $a^\dagger a$, because of the definition (1), the following equation is true:

$$f(aa^\dagger) - f(a^\dagger a) = 1 \tag{14}$$

this is a general deformation of the commutation relation (3), equation (14) also shows that the functions $f(x)$ (or $F(x)$) are the basic functions of the deformation theory and $g(x)$ is an auxiliary function. So we have proved the following general proposition.

Proposition 1. Let $f(x)$ be a real function and a, $a^\dagger$ two Hermitian conjugate operators satisfying the commutation relation

$$f(aa^\dagger) - f(a^\dagger a) = 1.$$

Then the operator

$$N = f(a^\dagger a)$$

satisfies the relations

$$[a, N] = a \qquad \text{and} \qquad [a^\dagger, N] = -a^\dagger.$$

Let $|\alpha\rangle$ a base of eigenvectors of the number opeator N

$$N|\alpha\rangle = \alpha|\alpha\rangle. \tag{15}$$

Equations (5) imply that the operator a (or $a^\dagger$) is a destruction (or a creation) operator such that

$$a|\alpha\rangle = \sqrt{[\alpha]}\,|\alpha - 1\rangle \qquad a^\dagger|\alpha\rangle = \sqrt{[\alpha + 1]}\,|\alpha + 1\rangle \tag{16}$$

where $[\alpha]$ is a function of α; from equation (1) we find that

$$[\alpha+1]=g([\alpha]) \qquad \text{or} \qquad f([\alpha+1])=1+f([\alpha]). \tag{17}$$

From these equations and the property (11) of the function $g(x)$, we conclude that

$$[\alpha]=F(\alpha). \tag{18}$$

The eigenvector $|0\rangle$, corresponding to the zero eigenvalue of the number operator N, satisfies the following:

$$\text{if} \qquad F(0)=0\,(\text{or } f(0)=0) \text{ then } a|0\rangle=0. \tag{19}$$

In this letter we assume that the function $F(x)$ is zero for $x=0$. The function $f(x)$ should be holomorphic around zero on the complex plane. In the case of the ordinary and the q-deformed oscillators

$$f'(0)\neq 0$$

therefore the inverse function $F(z)$ exists and it is holomorphic around zero. The above restriction is a sufficient condition but not a necessary one, because one can construct functions $F(z)$ which

(i) are well defined on the real positive axis, but not necessarily holomorphic around zero in the complex plane,

(ii) have an inverse function holomorphic around zero.

With this kind of function the proposed method can be applied, because the essence of the proposed constructions is based on the assumption that $f(z)$ is holomorphic, as we can see from equations (7)–(10). Even these restrictions are very strong ones, because one can construct examples where the method is applied and the functions $F(z)$ and $f(z)$ do not satisfy these constraints (an example is given below in equation (34)).

By induction the following proposition can be shown.

Proposition 2. The eigenvectors of the number operator $N=f(a^{\dagger}a)$ are generated by the formula

$$|n\rangle=\frac{1}{\sqrt{[n]!}}(a^{\dagger})^{n}|0\rangle \tag{20}$$

where

$$[n]!=\prod_{k=1}^{n}[k]=\prod_{k=1}^{n}F(k). \tag{21}$$

These eigenvectors are also eigenvectors of the energy operator

$$H=\frac{A}{2}(a^{\dagger}a+aa^{\dagger})$$

corresponding to the eigenvalues

$$E_n=\frac{A}{2}([n+1]+[n])=\frac{A}{2}(F(n+1)+F(n)). \tag{22}$$

An obvious identity is the following one:

$$a^\dagger a = F(N).$$

From this relation we can redefine the basic equation (1) as

$$aa^\dagger = g(F(N)) = F(N+1). \tag{23}$$

Therefore the deformed commutator is defined by

$$[a, a^\dagger] = F(N+1) - F(N). \tag{24}$$

The q-deformed oscillator is generated with this theory by putting

$$F(x) = \frac{\sin(\tau x)}{\sin(\tau)} \qquad q = \exp(i\tau). \tag{25}$$

Using this deformed oscillator we can construct the associate *deformed Lie algebras.* Let be $[X_a]_{ij}$ the fundamental representation of the algebra and a_i is a set of deformed oscillators then the operators:

$$\chi_a = \sum_{ij} a_i^\dagger [X_a]_{ij} a_j \tag{26}$$

are the deformed *generators* of the quantum deformation of the algebra [2, 3].

Let consider the deformed oscillator realization of SU(2):

$$\begin{aligned} &J_+ = a_1^\dagger a_2 \qquad J_- = a_2^\dagger a_1 \\ &J_0 = \tfrac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2) \qquad l = \tfrac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2) \end{aligned} \tag{27}$$

from this equation we find that

$$N_1 = a_1^\dagger a_1 = l + J_0 \qquad N_2 = a_1^\dagger a_1 = l - J_0. \tag{28}$$

After a little algebra the commutation relations of the deformed SU(2) can be calculated:

$$\begin{aligned} &[J_0, J_+] = J_+ \qquad [J_0, J_-] = -J_- \\ &[J_+, J_-] = H(l - J_0, l + J_0] \qquad [l, J_\pm] = 0 \qquad [l, J_0] = 0 \end{aligned} \tag{29}$$

where the function $H(x, y)$ is defined as

$$H(x, y) = F_1(x)F_2(y+1) - F_1(x+1)F_2(y). \tag{30}$$

If there is a *symmetric* SU(2), which means that the basic deformed oscillators are the same ones, i.e. $F_1(x) = F_2(x) = F(x)$, then the function $H(x, y)$ is antisymmetric:

$$H(y, x) = -H(x, y).$$

If $|0\rangle = |0\rangle_1 |0\rangle_2$ is the eigenvector of the fundamental level for both oscillators:

$$N_1|0\rangle = 0 \qquad N_2|0\rangle = 0$$

then using proposition 2 and equations (16) and (18) we can show the following quite general proposition.

Proposition 3. If the operators a_1 and a_2 satisfy the *commutation* relations:

$$f_i(a_i a_i^\dagger) - f_i(a_i^\dagger a_i) = 1 \qquad i = 1, 2$$

then the eigenstates of the deformed SU(2) operators (27) are given by the formulae

$$|j, m\rangle = ([j+m]_1! [j-m]_2!)^{-1/2} (a_1^\dagger)^{j+m} (a_2^\dagger)^{j-m} |0\rangle$$

where

$$[\alpha]_i = F_i(\alpha) = f^{-1}(\alpha) \qquad i = 1, 2.$$

One easily verifies that

$$\begin{aligned} J_+|j, m\rangle &= ([j+m]_1[j-m+1]_2)^{1/2}|j, m+1\rangle \\ J_-|j, m\rangle &= ([j+m+1]_1[j-m]_2)^{1/2}|j, m-1\rangle \end{aligned} \tag{31}$$

and that

$$J_0|j, m\rangle = m\,|j, m\rangle \qquad I|j, m\rangle = j\,|j, m\rangle. \tag{32}$$

Similar construction can be defined for the deformed SU(3) and the other deformed generalisations of the classical groups. The question of the co-multiplication of the SU(2) groups is under investigation. Equation (31) constraints the permitted values of j and m, because the following restrictions should be verified:

$$F_i(j \pm m) \geqslant 0.$$

The symmetric case $F_1(x) = F_2(x) = F(x)$ appears to be more interesting.

The abstract generalization of the proposition 3 is the following one.

Proposition 4. For the operator algebra

$$[J_0, J_+] = J_+ \qquad [J_0, J_-] = -J_-$$

$$[J_+, J_-] = H(I - J_0, I + J_0) \qquad [I, J_\pm] = 0 \qquad [I, J_0] = 0$$

where $H(x, y)$ is a real antisymmetric function of x, we can define a base $|j, m\rangle$ satisfying the relations

$$J_\pm|j, m\rangle = (F(j \pm m)F(j \mp m + 1))^{1/2}|j, m \pm 1\rangle$$

and

$$J_0|j, m\rangle = m|j, m\rangle \qquad I|j, m\rangle = j|j, m\rangle$$

where the function $F(x)$ is determined by

$$H(x, y) = F(x)F(y+1) - F(y)F(x+1) \qquad \text{and} \qquad F(0) = 0. \tag{33}$$

The function $F(x)$ seems to be determined uniquely from the function $H(x, y)$, if that is true, then proposition 4 should be a theorem useful for the construction of solutions of a wide class of algebras. Similar propositions for the deformed generalizations of other classical groups such as SU(N) are under investigation.

The q-deformed oscillator can be reproduced with the following choice of the structure functions:

$$F(x) = \frac{\sin(\tau x)}{\sin \tau} \qquad f(x) = \frac{1}{\tau} \sin^{-1}(x \sin \tau)$$

the associated g function is given by equation (4). With this choice one can reproduce the relations of [5].

The method exposed here is a generalization of [6] where the case

$$H(x, y) = h(x - y)$$

is considered.

Also one can construct other solvable models after choosing:

$$\begin{aligned} &F(x) = x^k \qquad f(x) = x^{1/k} \qquad g(x) = (1 + x^{1/k})^k \\ &H(x, y) = x^k(1+y)^k - y^k(1+x)^k. \end{aligned} \tag{34}$$

In this case we see that the function $f(x)$ is not necessarily holomorphic, but the proposed method can be applied.

A more complicated case is the following one:

$$F(x)=\frac{\operatorname{sn}(\tau x)}{\operatorname{sn}\tau} \qquad f(x)=\frac{1}{\tau}F(k, x\operatorname{sn}\tau)$$

$$g(x)=\frac{\operatorname{sn}(\tau+F(k, x\operatorname{sn}\tau))}{\operatorname{sn}\tau}$$

where $\operatorname{sn}(u)$ and $F(k, q)$ are elliptic functions and integrals.

An attractive application of the generalized deformed oscillator can be the search of the *equivalent* oscillators representing exactly a given energy spectrum E_n considered as a function of n. In this case from equation (22) we can find the appropriate $F(x)$ function and derive the equivalent deformed oscillator. Also the general theory, exposed here, gives solvable nonlinear deformed groups which can be used to describe broken symmetries.

Another interesting problem is the study of the solutions of the general equations (22) and (33), which determine the structure function $F(x)$ from an energy spectrum or from the structure function of a deformed algebra.

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